2. Conservation laws and basic equations

Equatorial region is mapped well by cylindrical (Mercator) projection. Corresponding to the local Cartesian (eastward, northward and upward) coordinates: (x, y, z)[5], the velocity vector is written as

$$\boldsymbol{u} \equiv (u, v, w) \equiv \left(\frac{Dx}{Dt}, \frac{Dy}{Dt}, \frac{Dz}{Dt}\right), \tag{2.1}$$

where t is time and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}, \quad \boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{2.2}$$

are the material (Lagrangian) time derivative and spatial derivative operator.

We start from three conservation laws (four equations): the equation of motion (Newton's 2nd law; angular momentum conservation law):

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} + \frac{1}{\rho} \boldsymbol{\nabla} p = -\boldsymbol{g} + \boldsymbol{F}; \tag{2.3}$$

the equation of continuity (mass conservation law)

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{u}, \quad \frac{Dr}{Dt} = -s + S; \tag{2.4}$$

the equation (the 1st law) of thermodynamics (entropy conservation law)

$$C_p \frac{D \ln \theta}{Dt} = \frac{J + Ls}{T}, \quad \theta \equiv T \left(\frac{p_{00}}{p}\right)^{R/C_p} \text{ (potential temperature)}; \tag{2.5}$$

where six variables are

$\boldsymbol{u}$, p (pressure), T (temperature), ρ (density), r (specific humidity), s (condensation ratio),

and parameters specified for physical properties of Earth and its atmosphere are the Coriolis parameter $\boldsymbol{f} \equiv (0, 0, f)$ with $f \equiv 2\Omega \sin \varphi$ (Ω:rotation angular velocity), gravity acceleration $\boldsymbol{g} \equiv (0, 0, g)$, gas constant for the dry air R, specific heat of constant volume for dry air C_p, latent heat for condensation (or sublimation) of water vapor L, and a standard pressure $p_{00} \equiv$1,000 hPa. External forcing terms (except for the gravity) are mechanical force $\boldsymbol{F} \equiv (F_x, F_y, F_z)$ such as friction and other momentum source, heating/cooling J and water vapor supply S.

Actual matter is a *continuum* which consists of innumerable molecules with:

$$\rho = \frac{\sum(\text{molecular mass})}{\text{unit volume}} = \frac{\text{molecular mass} \times \text{number}}{\text{unit volume}},$$

$$p = \frac{\sum(\text{molecular momentum})}{\text{unit area} \times \text{unit time}} = \frac{\text{molecular force}}{\text{unit area}},$$

$$T = \frac{\sum(\text{molecular kinetic energy})}{\text{molecular number} \times \text{Boltzmann constant}}.$$

Theoretical concept *fluid* corresponds to gas and liquid phases of actual matter such as atmosphere and ocean. Solid phase such as Earth crust is described by concepts of plastic, elastic or rigid. Based on these considerations we may add one more equation for thermodynamic state (the Boyle-Charles low for ideal gas; cf. Chapter 3)

---

[5]Relationships with longitude λ and latitude φ are

$$dx = a \cos \varphi \cdot d\lambda, \quad dy = a \cdot d\varphi.$$

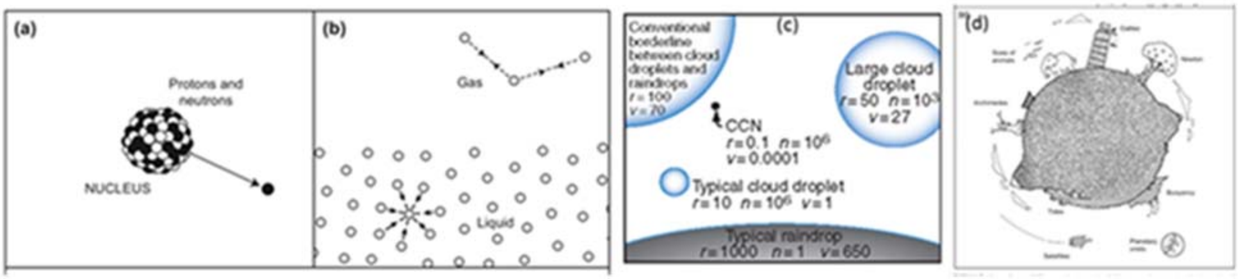

Fig. 2.1 Forces act in the nature. (a) Strong/weak nuclear forces, (b) electromagnetic intermolecular force and (d) gravitational force (Israelachvili, 1992). In the atmosphere (c) cloud and precipitation processes with intermediate scales in which both (b) and (d) act are also important (Wallace & Hobbs, 2006).

$$p = R\rho T_v\,, \quad T_v \equiv T\left[1 + \left(\frac{m_d}{m_v} - 1\right)r\right] \text{ (virtual temperature);} \tag{2.6}$$

where additional parameters m_d and m_v are molecular masses of dry air and water vapor, respectively.

Forces (interactions) define and are dependent upon scales (Fig. 1). Strong and weak nuclear forces act only in the scales of an atomic nucleus and its constituents (elementary particles). The electromagnetic intermolecular force defines a phase change, that is *saturation* in homogeneous nucleation. Kelvin's curvature effect concerns a supercooled tiny droplet. Raoult's solute effect generates vapor pressure or boiled point depression of a solution from the pure water. Henry-Dalton's partial pressure law holds for a mixed gas. On the unsaturated surface of a droplet molecular diffusion caused evaporation.

Condensation at a solid surface (heterogeneous nucleation) makes a large droplet. If the solid is as huge as a planet, gravitational force acts to trap liquids or gases as satisfying phase equilibrium and density stratification. Water is also produced photochemically and erupted volcanically from the interior. On the other hand water is photodissociated, and resulting hydrogen and oxygen are separated gravitationally and lost from the atmosphere through escape to the space and oxidation of the solid planet, respectively. Earth's oceans have been maintained by a balance between those competing effects. In this maintenance the most important mechanism is the cloud-precipitation (coalescence/sublimation) processes, in particular near the very cold tropical tropopause as a result of radiative-convective equilibrium returning all water to the ground (Chapter 3). Clouds are generated more actively by upward motion of convection which may be caused by an instability due to latent heating. However, the latent heat is a result of cloud condensation/sublimation, and any external forcing (such as larger scale motion, orography or sea-land contrast) is necessary to cause such a *conditional* instability (Chapter 6).

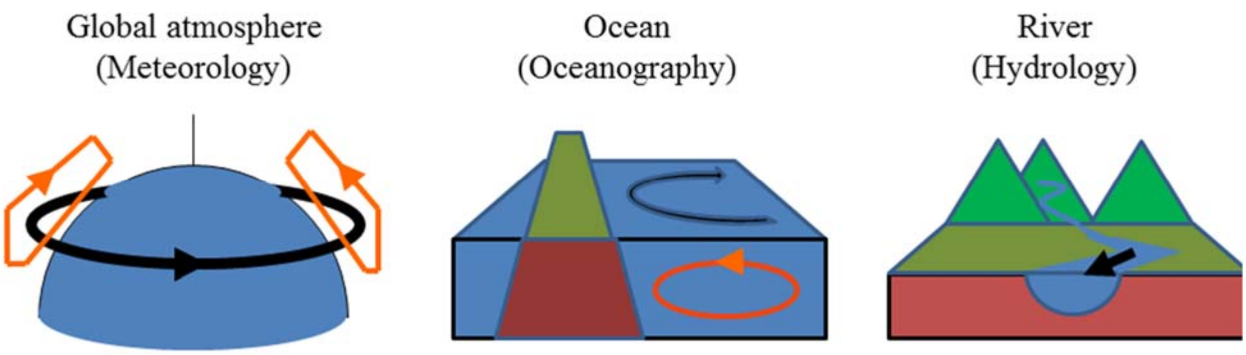

Fig. 2.2 Geophysical fluids.

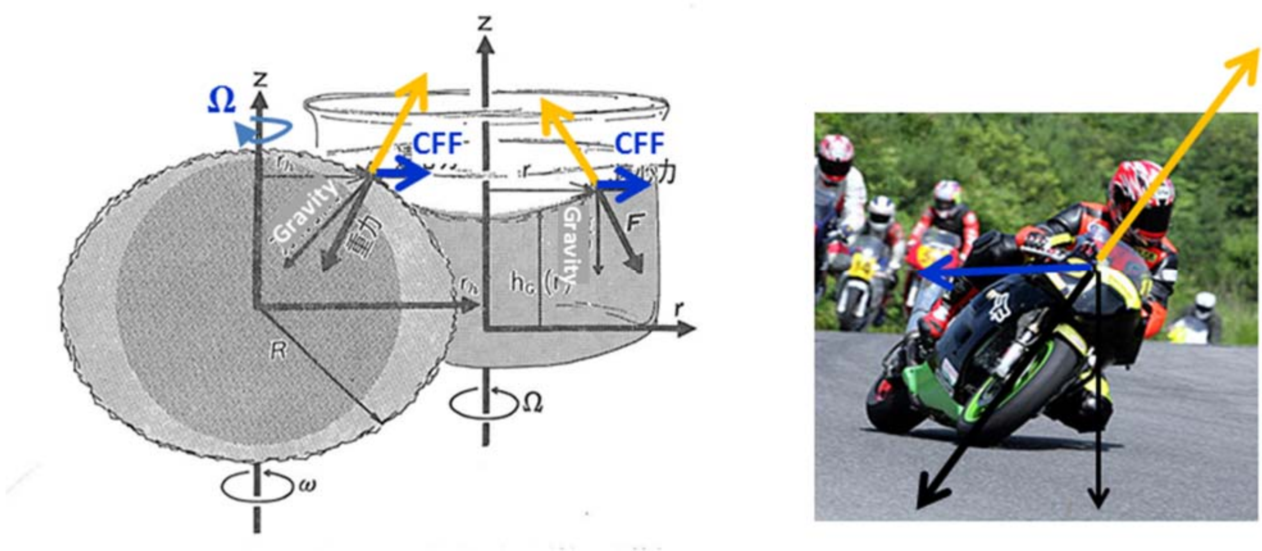

Fig. 2.3 Centrifugal force appearing generally in case of rotating or curving motion: (left) Earth; (middle) rotating water in a cup.; and (right) bicycle on a curved road.

The geophysical fluids or geofluids, such as atmosphere, oceans and rivers (Fig. 2.2), have almost common laws (2.3)-(2.5). The global atmosphere is compressible (as described by an equation of state to be mentioned later), almost closed, rather zonally homogeneous, and almost free except for the bottom boundary. The ocean is rather incompressible (with thermal expansion), almost closed, rather horizontally homogeneous, and restricted by the bottom and coastlines. The river is also almost incompressible, but is almost one-dimensional with complex boundaries.

A more generalized form of the equation of motion (2.3) involves a centrifugal force which arrears in general in case of a rotating or curving motion (Fig. 2.3): for a unit mass,

$$\text{centrifugal force} = \text{radius} \times (\text{angular velocity})^2 = (\text{velocity})^2 / \text{radius},$$

because velocity = radius × angular velocity. A bicycle rider on a curved road feels a vector sum of the gravity and the centrifugal force, by which a tire pushes the road slantwise. It must be balanced with a normal reaction of the road pushing the tire just with the same strength in the opposite direction, which is achieved by the rider inclining the bicycle. Similarly on a rotating water surface a modified gravity (working on the air above the surface) pushes the water slantwise, which must be balanced with an inclined pressure of the water working on the pitted surface.

Another example is Earth's or any planetary revolution on a circular orbit. For us on Earth we observe our Earth being always at a distance from Sun, just as a static balance between the solar gravitational force and a centrifugal force with the Earth's revolution around Sun (Fig. 2.4(b)). Alternatively, if we observe it from the space outside the solar system, Earth is accelerated (just as falling toward Sun) by the solar gravitation working always Sunward and perpendicular to Earth's revolution velocity (Fig. 2.4(a)). Similarly, for a body in fact resting on the Earth's surface, or rotating with the same velocity as the Earth's surface, we may observe it being accelerated always by an imbalance between the Earth's gravity (toward the Earth's center) and a normal force reacted from the Earth's surface (Fig. 2.4(c)). Alternatively, there is a static balance among the Earth's gravitation, the normal force and a centrifugal force with the rotation of the body which is the same as the Earth's surface (Fig. 2.4(d)), and we usually define the Earth's

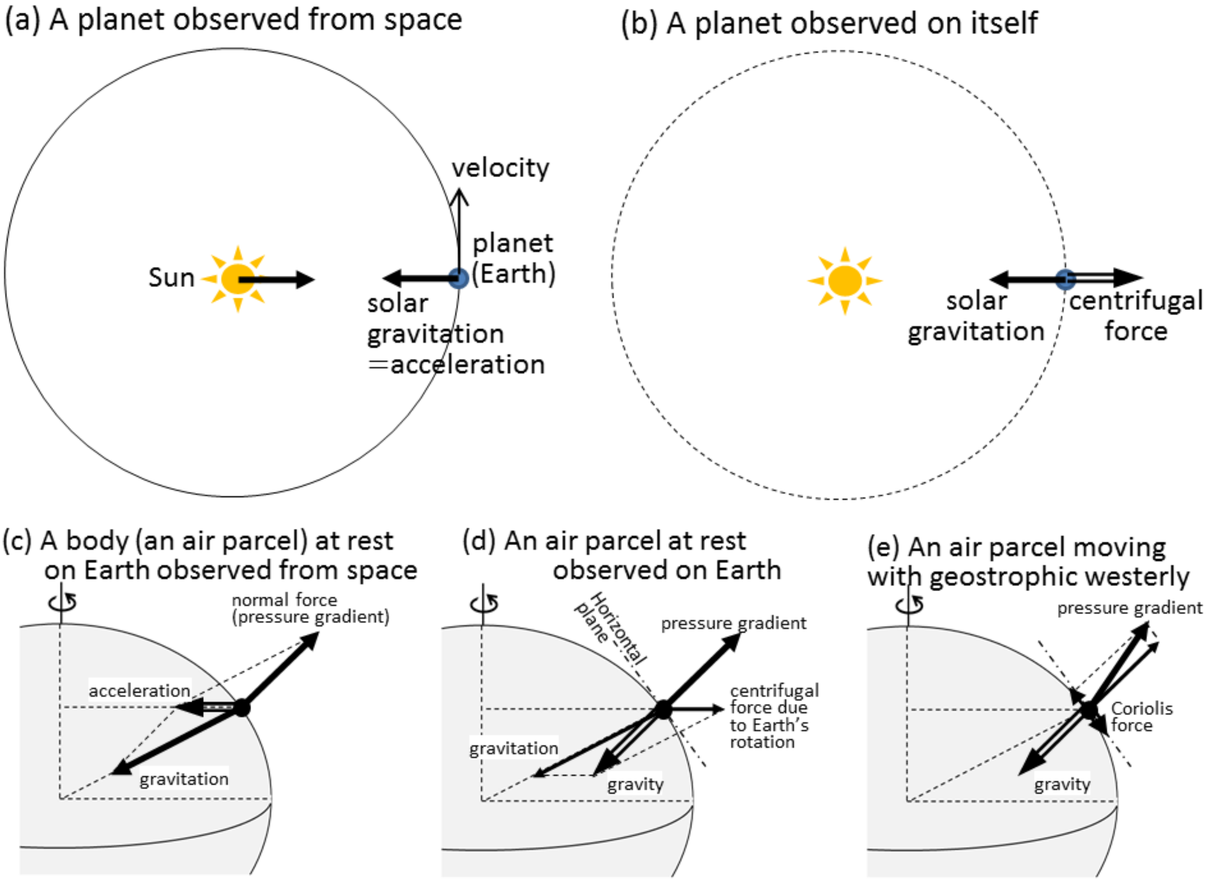

Fig. 2.4 Centrifugal and Coriolis forces. (a) Absolute and (b) relative views of a planet motion, and the latter involves the centrifugal force. (d) Absolute and (b) relative views of a body (meteorologically saying, an air "parcel") on Earth, and the latter involves the centrifugal force. (e) Earth-referenced view of a material point with an eastward (westerly) motion (superrotation) relative to Earth.

gravity by the sum of the gravitational force and the centrifugal force. Therefore the gravity by this definition is not exactly constant (maximum at the poles and minimum on the equator) nor toward the Earth's center, and the Earth's "horizontal" surface is not exactly spherical but *ellipsoidal*, which is similar to the pitted surface of rotating water (Fig. 2.3). The normal force corresponds to a pressure gradient force in rotating fluid dynamics or in meteorology.

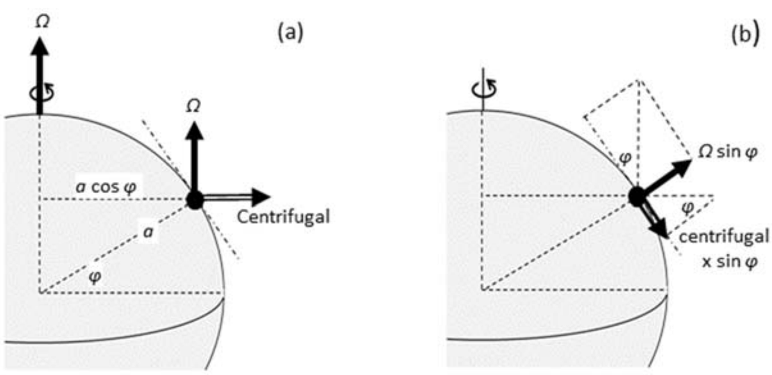

Fig. 2.5 (a) Centrifugal force generated by rotation, and (b) a horizontal centrifugal force generated by a vertical rotation.

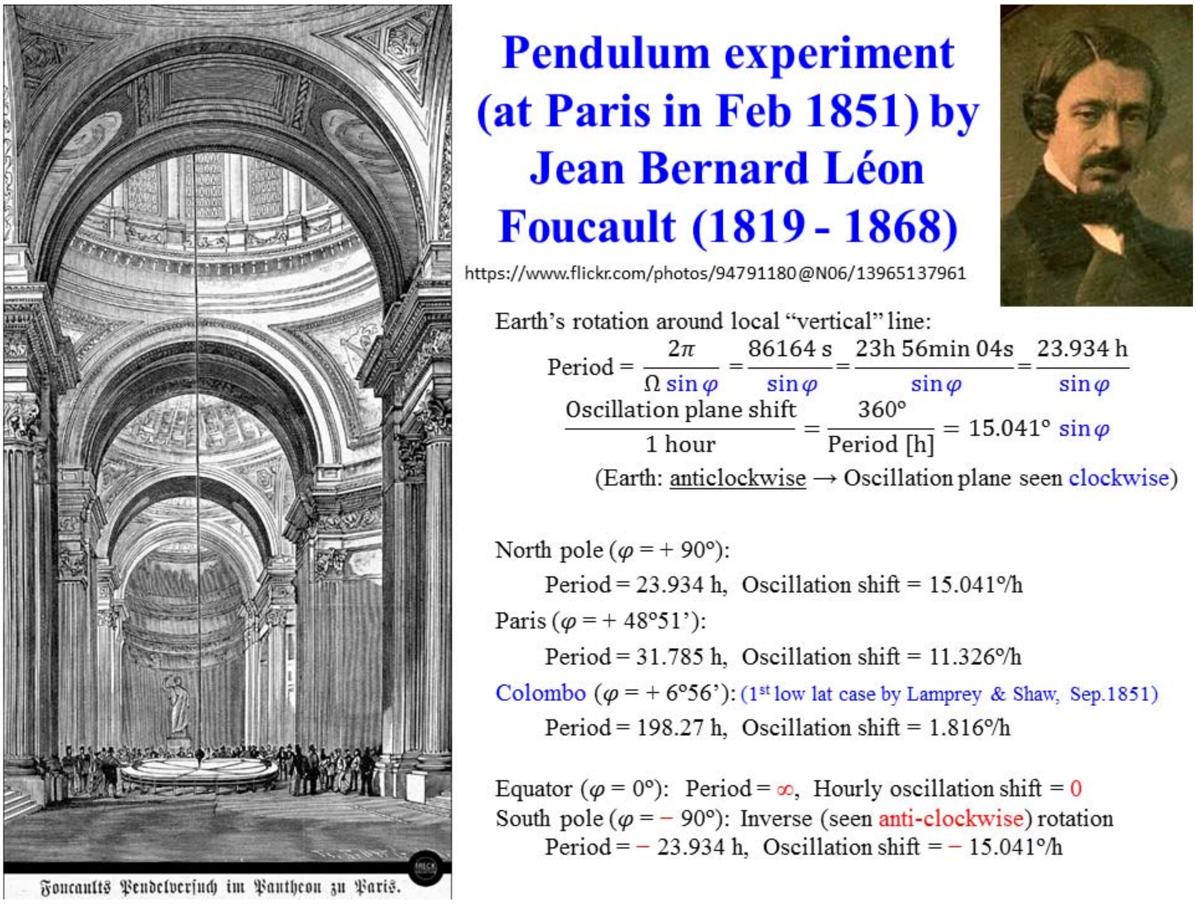

Fig. 2.6 Foiucault's pendulum experiment made people believe Earth's rotation completely. Within only a half year (which was very surprising at that time) similar experiments were carried out in many places, including low latitude region of which the first place was Colombo, Ceylon (Lamprey and Shaw, 1851).

If the body or parcel is in an eastward (westerly) motion u relative to Earth (Fig. 2.4(e)), the horizontal component[6] (multiplying $\sin\varphi$) of the centrifugal force becomes

$$\text{rotating radius } \times (\text{Earth's angular velocity + relative angular velocity})^2 \ \times \sin\varphi$$

$$= a\cos\varphi\left(\Omega + \frac{u}{a\cos\varphi}\right)^2 \sin\varphi \ = \ \frac{1}{2}a\Omega^2 \sin 2\varphi + (2\Omega\sin\varphi)\cdot u + \frac{u^2}{a}\tan\varphi,$$

of which the second term is the Coriolis force included as the second term fu of (2.3). The existence of the Coriolis force and its dependence on sin latitude were first proved by the Foucault experiment (Fig. 2.6). The first term is a part of the centrifugal force appearing already in Fig. 2.4(d), and is regarded to be included in g as its meridional dependency taking the maximum of about 1/300 of g at $\varphi = \pm\pi/4 =$45ºN/S, which implies that the ellipticity or deference between equatorial and polar radii is about 1/300. The last term is the centrifugal force by the relative motion, which is considerable only when u is quite large in a small radius $\ll a$ (such as in a tropical storm or in a tornado; Chapter 6), otherwise the planetary rotation Ω is quite small such as in Venus or Titan.

The continuity equation (2.4) was formulated at first by a Swiss mathematician-physicist Euler (Fig. 2.7). If the

[6]Neglect of the vertical component dependent on $\cos\varphi$ is called "traditional approximation" in meteorology, and is valid surprisingly even in the tropics with small φ, because the stratification is strong enough (cf. Phillips, 1966; Gill, 1982, §7.4).

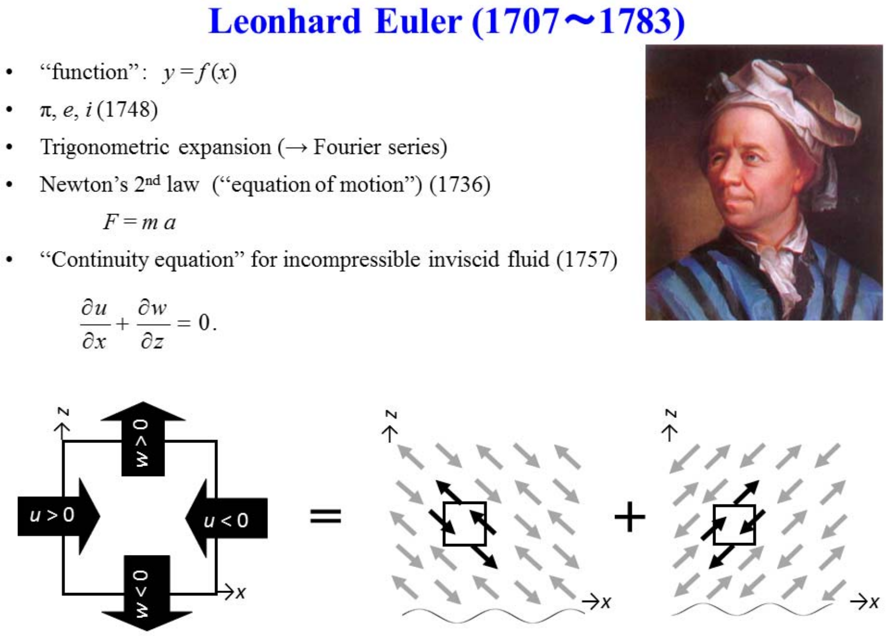

Fig. 2.7 Leonhard Euler and his achievements which are used even now. The portrait is by E. Handmann in 1753, obtained from https://en.wikipedia.org/wiki/File:Leonhard_Euler.jpg. The bottom panels show the simplest (two-dimensional non-divergent) form of his “continuity equation”.

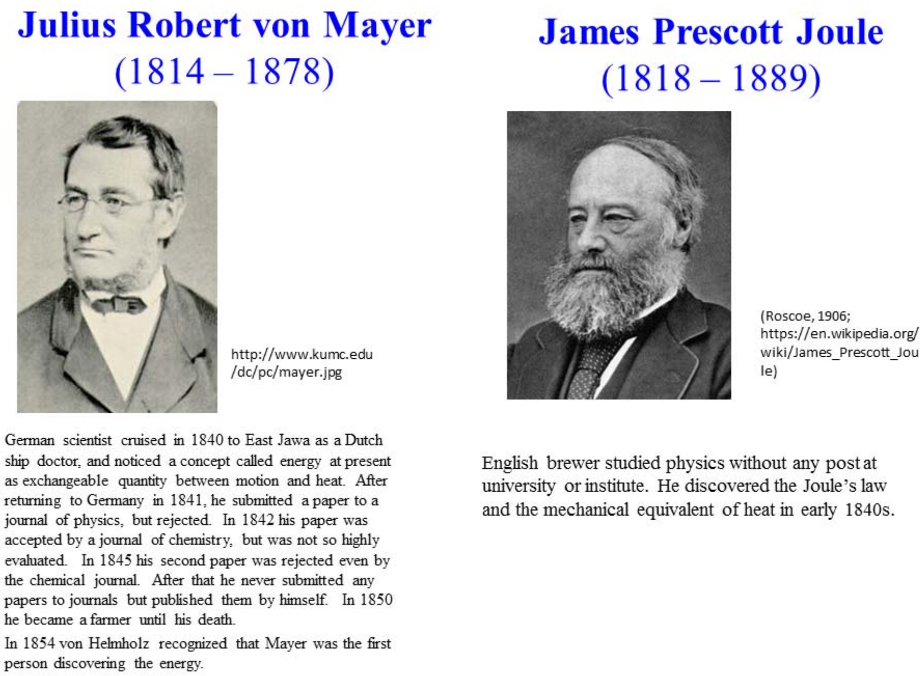

Fig. 2.8 Mayer and Joule. They and many others contributed to establishment of thermodynamics.

fluid is incompressible ($\rho =$ cons.), (2.4) becomes a nondivergent equation. Its two-dimensional case can be solved simply by a summation of two plane waves (see Section 6.1). In (2.4) continuity equations both for atmosphere and water vapor are shown.

The thermodynamic equation (2.5) was concerned by many scientists. Charles and Helmholtz will appear later. Mayer and Joule (Fig. 2.8) invented an important concept “energy” connecting all the physical laws (2.3)-(2.5), and the former obtained his idea in Dutch East Indies, namely Indonesia at present.

In conclusion of this chapter, the four (one vector and three scalar) equations (2.3) – (2.6) are closed (able to be solved mathematically) for the four (one vector and three scalar) variables $\boldsymbol{u}$, p, T and ρ, if the atmosphere is *dry* (without water vapor: $r = 0$, then the second equation of (2.4) is omitted). When water vapor is included but does not make phase change ($s = 0$, and no latent heating in (2.5)). the five equations including the second one of (2.4) are closed for the five variables including r. If the atmosphere is *moist* (with phase changes of water vapor, six equations with the addition of the Clapeyron-Clausius equation:

$$r \leq r_s \equiv \frac{(m_v/m_d)e_s}{p-(1-m_v/m_d)e_s} \approx \frac{m_v}{m_d}\frac{e_s}{p}\ , \quad \frac{de_s}{dT} \approx \frac{Le_s}{(m_d/m_v)RT^2} \tag{2.7}$$

are closed for the six variables including s. Even in the tropics where the moist process is essentially important, most of basic dynamical features described in Chapters 3-5 may be explained by simplified versions of the dry system (Fig. 2.9). The most process will be mentioned mainly concerning cloud systems in Chapter 6.

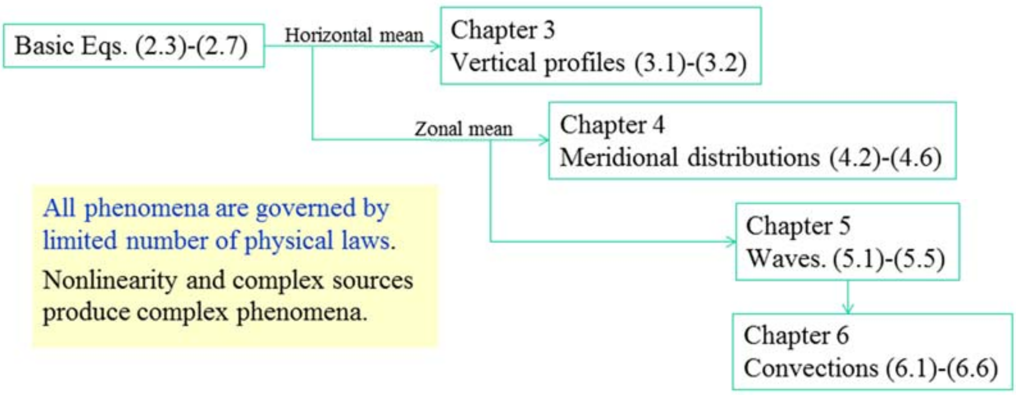

Fig. 2.9 Governing equations used in this book.

Exercise 2

(1) How different the Coriolis force in the northern and southern hemispheres? How about at the equator?

(2) Do you think the motorcycle rider and the bathtub vortex must feel the Coriolis force of the earth’s rotation? How about the Coriolis for the solar system, or of galaxy?

Answers:

(1) sinφ changes sign and direction of Coriolis force becomes opposite in northern/southern hemispheres . It becomes 0 and Coriolis force vanishes at the equator.

(2) No, because the earth’s rotation is in the time scale of 1 day. Similarly, we can neglect the earth’s revolution around the sun with 365 days >> 1 day, as well as the solar revolution in our galaxy with 20 million years.