3. Atmospheric vertical structure: Radiative-convective equilibrium

3.1. Radiative equilibrium

The vertical variability due to gravity is superior to and any other variabilities including tropical specialty. We shall review briefly the vertical one-dimensional ($\partial/\partial x = \partial/\partial y = 0$), steady ($\partial/\partial t = 0$) and rest ($\boldsymbol{u} = 0$; but may involve small random eddies) case. Then the vertical component of the equation of motion (2.3) becomes

$$\frac{1}{\rho}\frac{dp}{dz} = -g, \tag{3.1}$$

which is called *hydrostatic equilibrium*. The equation of thermodynamics (2.5) becomes

$$-\frac{C_p}{\theta}K\frac{d^2\theta}{dz^2} = \frac{J + LS}{T}, \tag{3.2}$$

where $s = S$ from the second equation of (2.4) under the steady rest condition, and the so-called eddy diffusion[7] $-K\,\partial^2\theta/\partial z^2$ is inserted for omitting the vertical advection term $w\,\partial\theta/\partial z$.

The radiative heating term is assumed as

J = absorption of I ("net" upward infrared radiation flux (per unit time and unit area))
+ absorption (conduction) of sensible heat flux at land/sea surface.

As the bottom boundary condition, we specify that

the second term above + LS = I_s (solar radiation at land/sea surface).

Then (3.2) may be rewritten as

$$-C_pK\frac{d^2\theta}{dz^2} = \frac{\theta}{T}\left(-\frac{1}{\rho}\frac{dI}{dz}\right) \tag{3.3}$$

The simplest case is that the atmosphere is completely rest (with no eddies, i.e. $K = 0$). Then we have

$$\frac{dI}{dz} = 0, \tag{3.4}$$

which implies that the net radiation is emitted to the space without any additional source/sink, and is called *radiative*

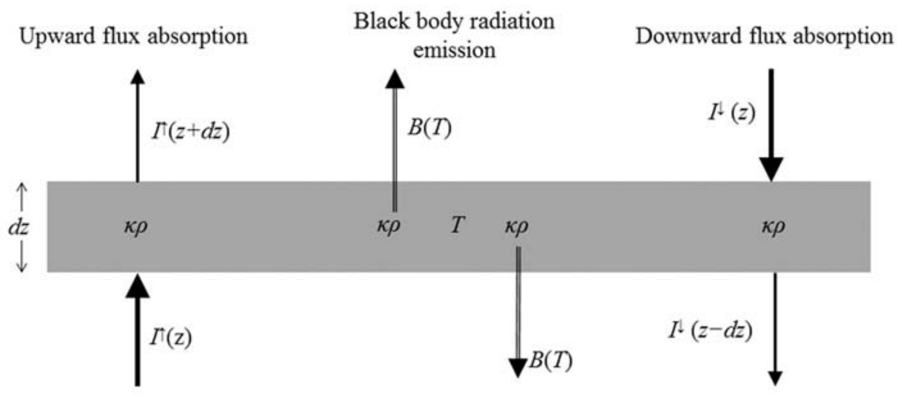

Fig. 3.1 Two stream approximation in a gray body atmosphere.

[7]The advection term may be rewritten as $\boldsymbol{u}\cdot\boldsymbol{\nabla}\theta = \boldsymbol{\nabla}\cdot\theta\boldsymbol{u} - \theta\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{\nabla}\cdot\theta\boldsymbol{u}$, because (2.4) is $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ under the steady condition. By the horizontal homogeneity, $\boldsymbol{\nabla}\cdot\theta\boldsymbol{u} = \partial\,\theta w/\partial z$. The vertical flux is assumed as being counter-gradient (from higher θ to lower θ, namely homogenizing θ): $\theta w \equiv -K\,\partial\,\theta/\partial z$, where K is called *eddy diffusivity*.

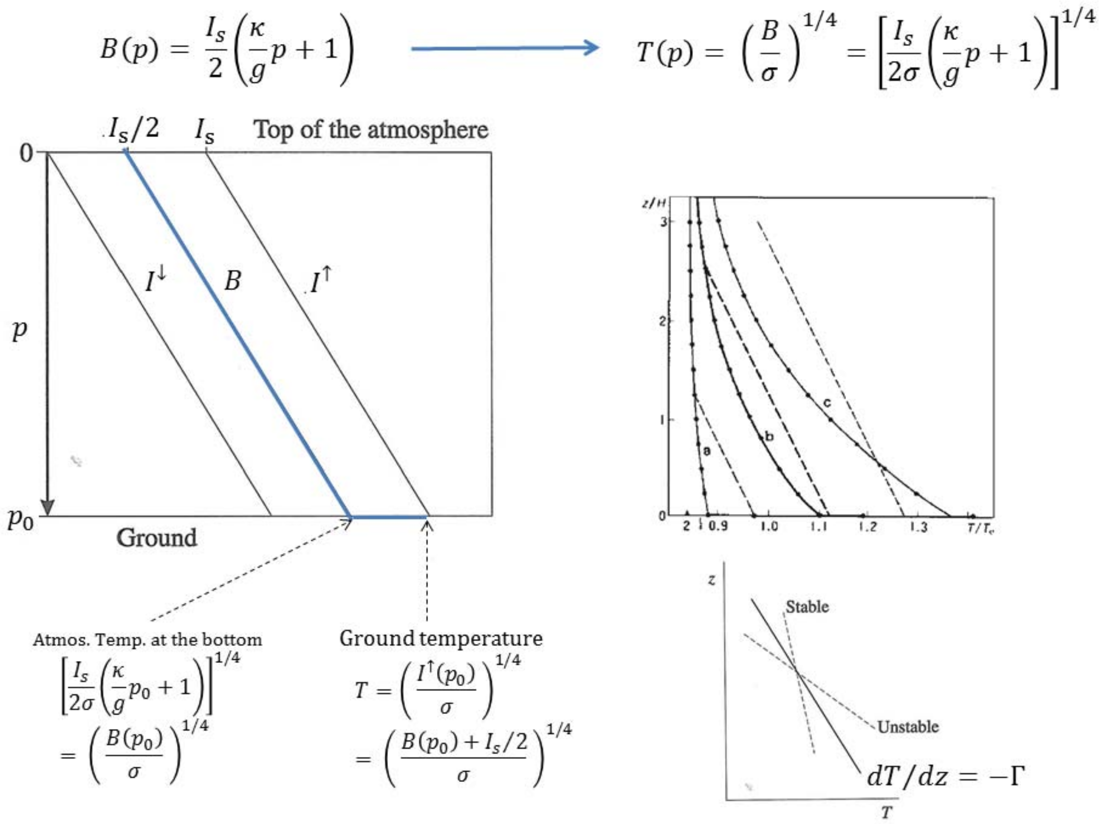

Fig. 3.2 Radiative equilibrium temperature profile in p-T^4 space with a gap at the ground (left) and in z-T space with convectively neutral adjustment in a layer above the ground (right),

equilibrium. Usually (3.4) is solved under "gray body" and "two stream" approximations: the net radiation is approximated by upward and downward fluxes, $I^{\uparrow}$ and $I^{\downarrow}$, such that

$$I \equiv I^{\uparrow} - I^{\downarrow},$$

and "greenhouse gases" in the atmosphere absorb each of them by a factor parameterized as $\kappa\rho$ (per unit volume).and re-emit black body radiation $B = \sigma T^4$ given by the Stefan-Boltzmann law (Chapter 1). Namely[8],

$$\frac{dI^{\uparrow}}{dz} = \kappa\rho\left(-I^{\uparrow} + B\right), \quad \frac{dI^{\downarrow}}{-dz} = \kappa\rho\left(-I^{\downarrow} + B\right),$$

where the emission rate equals to the absorption rate as stated by the Kirchhoff law. Taking sum and difference of these equations, substituting $dz = -dp/g\rho$ from (3.1), and using (3.4), we have

$$0 = \frac{g}{\kappa}\frac{dI}{dp} = \left(I^{\uparrow} + I^{\downarrow}\right) - 2B, \quad \frac{g}{\kappa}\frac{d\left(I^{\uparrow} + I^{\downarrow}\right)}{dp} = I \equiv I_s.$$

The former gives $\left(I^{\uparrow} + I^{\downarrow}\right) = 2B$, and the latter becomes

$$\frac{dB}{dp} = \frac{\kappa}{2g}I_s.$$

[8]In a little bit more correct calculation, integration of flux with cos zenith gives a factor 2/3 for $dI^{\uparrow}/dz$ and $dI^{\downarrow}/(-dz)$ and π for B, and these factors are deleted by replacing p by so-called "optical depth". However, results are essentially the same as below.

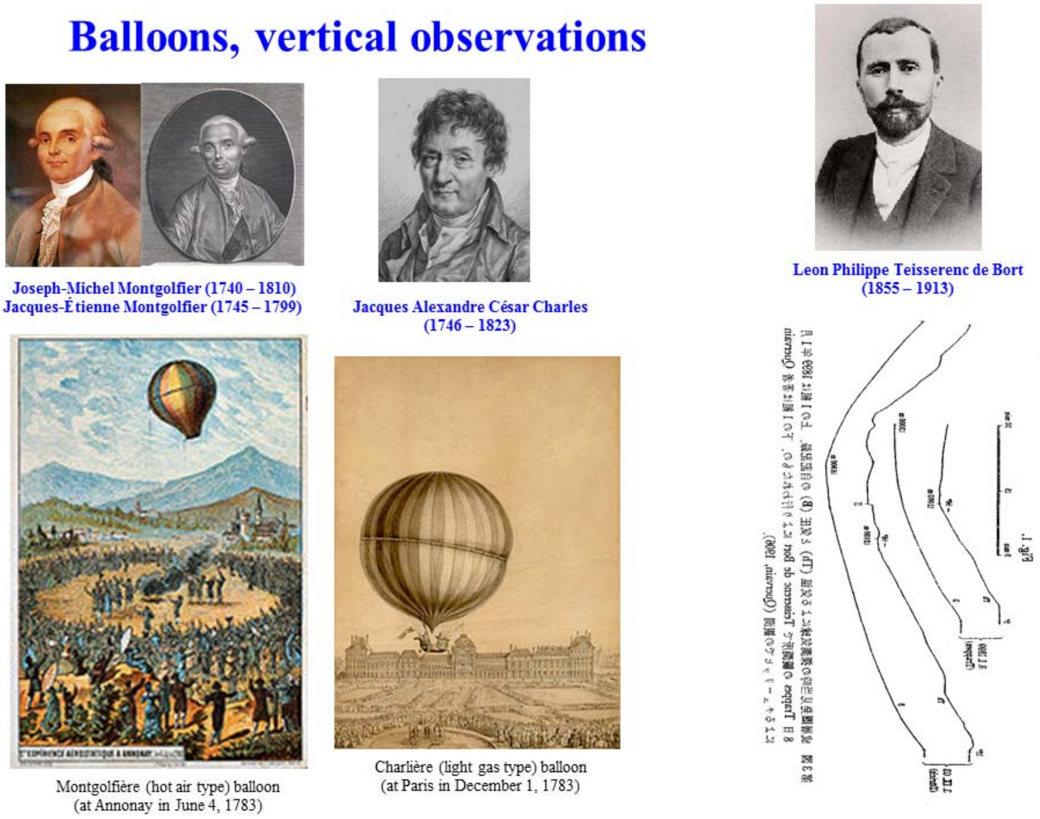

Fig. 3.3 The pioneers of two types balloons: Montgolfier brothers (https://en.wikipedia.org/wiki/Montgolfier_brothers) and Charles (https://en.wikipedia.org/wiki/Jacques_Charles); and balloon-borne thermometer-barometer records (in a Japanese document by T. Matsuno, 1980) by Teisserenc de Bort (https://web.archive.org/web/20120315112414/http://www.meteo.fr/meteo_france/implantation/setim/historique/historique.html) and others.

This equation is solved, with a top boundary condition:

$$I^{\downarrow} = 0\,, \quad I = I^{\uparrow} = I_s \qquad \text{at} \quad p = 0,$$

of which the solution is

$$T(p) = \left(\frac{B}{\sigma}\right)^{1/4} = \left[\frac{I_s}{2\sigma}\left(\frac{\kappa}{g}p + 1\right)\right]^{1/4} \qquad \left(I^{\uparrow} = B + \frac{I_s}{2}\,, \quad I^{\downarrow} = B - \frac{I_s}{2}\right). \tag{3.5}$$

(see the left panel of Fig. 3.2).

The temperature profile of the radiative equilibrium (3.5) decreases upward almost exponentially in the geometric vertical coordinate (altitude z), because p decreases upward exponentially. At the top we have

$$T(0) = (I_s/2\sigma)^{1/4} \approx 0.84 \cdot (I_s/\sigma)^{1/4},$$

and the "equivalent black-body temperature" (obtained in Chap.1 as $I_s = S_0$) is obtained at

$$p = g/\kappa.$$

At the bottom (atmosphere above ground, $p = p_0$), we have

$$T(p_0) = \left[\frac{I_s}{2\sigma}\left(\frac{\kappa}{g}p_0 + 1\right)\right]^{1/4} = \left(\frac{B(p_0)}{\sigma}\right)^{1/4},$$

which is cooler than the ground temperature:

$$T_s = \left(\frac{I^{\uparrow}(p_0)}{\sigma}\right)^{1/4} = \left(\frac{B(p_0) + I_s/2}{\sigma}\right)^{1/4},$$

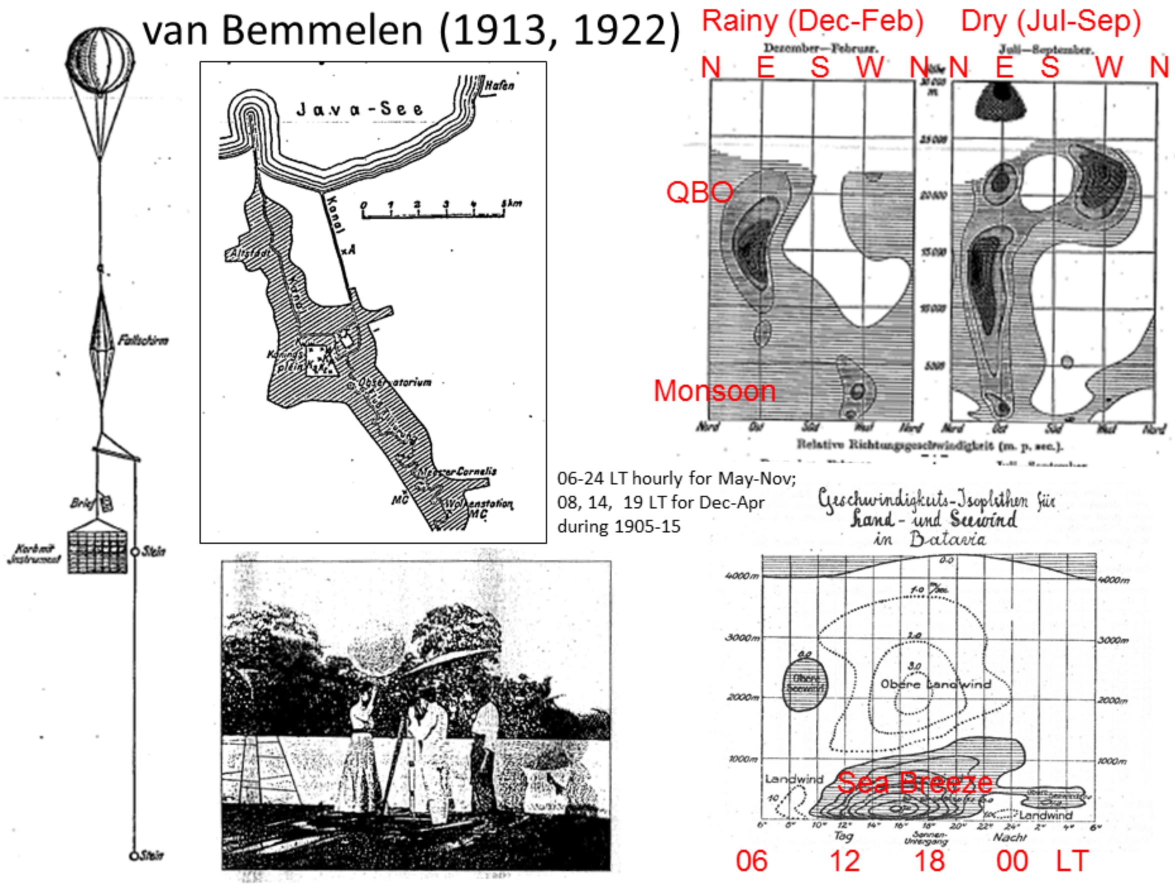

Fig. 3.4 The pioneering vertical meteorological sounding (and many other fields) in the tropics or equatorial region by van Bemmelen (1913, 1922) in Batavia (Jakarta at present).

because the ground must emit the upward radiation $I^{\uparrow}(p_0)$. This temperature *gap* with an infinitely large temperature gradient ($dT/dz = -\infty$) is due to neglection of any other heat transport processes, i.e., conduction or convection.

In sufficiently higher altitudes $I^{\downarrow} \ll I^{\uparrow} \approx I^{\uparrow}(0)$, so the net heating J in (3.2) or (2.5) is governed almost only by the upward flux $I^{\uparrow}$:

$$J \approx \kappa\left(I^{\uparrow} - 2B\right) \approx -C_p\alpha\,(T - T_R)\,, \quad \alpha \equiv \frac{2\kappa}{C_p}\left(\frac{\partial B}{\partial T}\right)_{T=T_R}, \tag{3.6}$$

where T_R is the radiative equilibrium temperature and α is called the Newtonian cooling coefficient, because Newton considered a temperature transport proportional to the temperature difference.

3.2. Radiative-convective equilibrium

The vertical temperature gradient of the radiative-equilibrium temperature profile (3.5) becomes larger in the lower altitudes as the atmosphere is deeper[9], and is always (even in a shallow atmosphere) infinitely large at the bottom. Therefore the lower atmosphere causes convective (Rayleigh-Taylor) instability:

$$\frac{d\theta}{dz} < 0, \quad \text{i.e.}\ \frac{dT}{dz} < -\Gamma \quad \left(\Gamma \equiv \frac{g}{C_p} \approx 10\ \text{K/km}\right) \qquad \text{or} \qquad \frac{d\ln T}{d\ln p} > \frac{R}{C_p}$$

[9](3.5) gives $d\ln T/d\ln p = 1/[4(1 + g/\kappa p)] < 1/4 \le R/C_p \approx 2/5,\ 2/7,\ 1/4$ for 1-, 2-, poly-atom molecule, thus only barely stable.

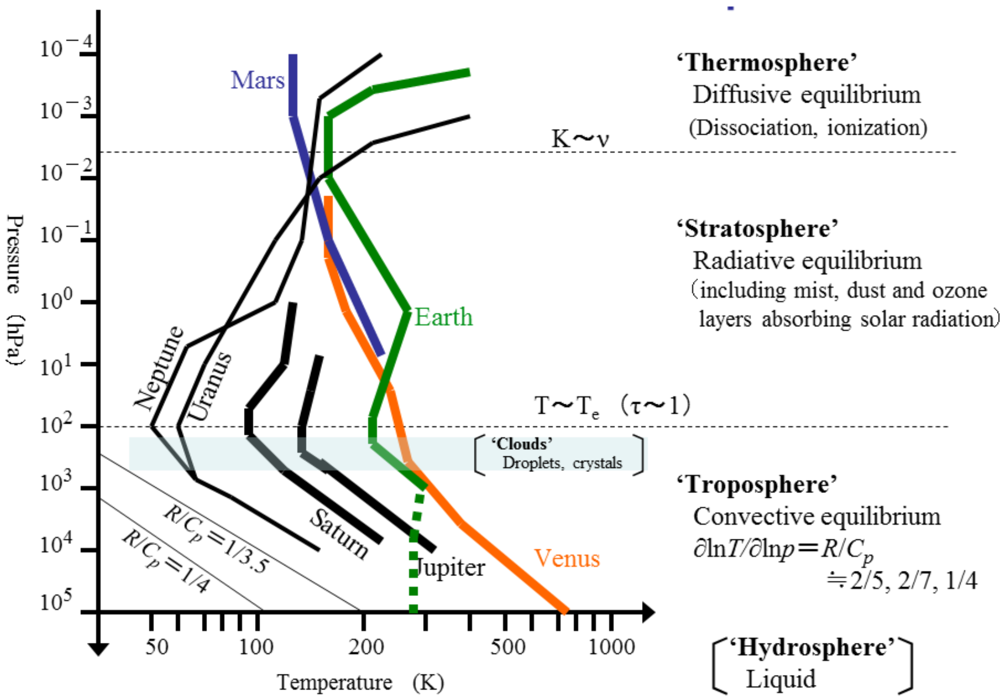

Fig. 3.5 Vertical temperature profiles of planetary atmospheres.

where Γ is called the *dry adiabatic lapse rate*. In order to avoid this instability within this one-dimensional problem, we need to take a sufficiently large eddy diffusivity ($K \to \infty$), and replace (3.5) in the unstable layer by a neutral situation expected as the ultimate result of eddy diffusion:

$$\frac{d\theta}{dz} = 0 \quad \text{i.e.} \quad \theta = \text{const.}\,, \; T = T(z=0) - \Gamma z \tag{3.7}$$

(see the right panel of Fig. 3.2).

Actually, as will be mentioned in the next section, water vapor included in the atmosphere may cause phase changes and latent heating, which modified the lapse rate from the dry value Γ. In a realistic numerical model, Γ is replaced by an observed mean lapse rate $\Gamma_c \approx 6.5$ K/km, and the so-called *convective adjustment* is coded:

If $dT/dz < -\Gamma_c$ is obtained, it is replaced by $dT/dz = -\Gamma_c$

(Manabe and Strickler, 1964). By this adjustment values of T at the top and bottom of the unstable layer are respectively increased and decreased from those given by the radiative equilibrium, as the result of upward heat transport by convection[10]. Therefore, we have a corrected temperature profile:

$$T(p) = \begin{cases} \left[\frac{I_s}{2\sigma}\left(\frac{\kappa}{g}p_{tp}+1\right)\right]^{1/4} \cdot \left(\frac{p}{p_{tp}}\right)^{\Gamma_c R/g} & \text{for the lower layer } (p > p_{tp}) \\ \left[\frac{I_s}{2\sigma}\left(\frac{\kappa}{g}p+1\right)\right]^{1/4} & \text{for the upper layer } (p \le p_{tp}) \end{cases}, \tag{3.8}$$

[10]The altitudes of the top and bottom and their values of T are determined so as to conserve the energy before and after the adjustment.

which is called radiative-convective equilibrium. The lower and upper layers are called *troposphere* and *stratosphere*, respectively, and their boundary level ($p = p_{tp}$) is called *tropopause*. The tropopause temperature is also written as

$$T(p_{tp}) = T(p_0) - \Gamma_c \cdot z(p = p_{tp}).$$

If the surface temperature $T(p_0)$ is increased (warming) with keeping Γ_c, the tropopause (still on the radiative equilibrium profile decreasing upward slowly) becomes higher and cooler. This explains qualitatively why the tropical tropopause is higher and cooler than the polar tropopause.

Historically French engineers/scientists in 18-19 centuries contributed greatly to development of balloons and discovery of the stratosphere (Fig. 3.3) There are (even now) basically two types of balloons: Montgolfier's hot air type, and Charles' light gas type. The former is now used almost only for sports in the lower troposphere, whereas the latter (using hydrogen or helium) is used for meteorological observations and other scientific ballooning. Charles is known also as the great chemico-physicist who discovered the Boyle-Charles law (2.6). A century after these developments, also after many (more than 200) balloon launches, Teisserenc de Bort published the first paper in 1902 to conclude existence (and also to give names) of troposphere and stratosphere. In the same era a German meteorologist Assmann (who also developed a standard thermometer-hygrometer) discovered and published the same evidence independently. Just after these discoveries van Bemmelen (1913, 1922) started balloon observations and confirmed essentially the same structure of (with higher and cooler) tropopause over the tropics (Fig. 3.4). Actually the temperature turns abruptly to upward increase in the stratosphere, because ozone absorbs the ultraviolet component involved in the solar radiation (Section 4.4), which is a peculiar feature of Earth (Fig. 3.5) in spite that the tropopause-stratosphere structure is common in all the planets having stable atmospheres.

3.3. *Moisture effect*

If the atmosphere is purely of water vapor at largest amount ($p = e_s$), using the hydrostatic equation (3.1) for this atmosphere, the Clapeyron-Clausius equation (2.7) becomes[11]

$$\frac{dT}{dz} = -\frac{g}{L/T} \equiv -\Gamma_s., \qquad (3.9)$$

This profile is driven without using any other equation such as the thermodynamic equation (2.5), and must be an asymptote for all the profiles given for example by including a radiative forcing J. If the bottom has a liquid water (ocean) or if a lower layer above the bottom is saturated, (not gradient but) temperature at the bottom or the top of lower layer is determined by (2.7). This implies that, if the external heating (solar radiation) is larger than that given by (2.7), there is no longer an radiative or radiative-convective equilibrium temperature profile (3.8), which is called "runaway greenhouse effect" and is considered to have made Venus lose ocean (Nakajima et al., 1992)[12].

For the present Earth's atmosphere, phase-changeable constituent (water) is not a major constituent ($p \gg e_s$)[13]. Evaporation occurs almost always at the bottom (both oceans and lands), and condensation (or sublimation) occurs

[11]The gas constant for water vapor is $\frac{Nk}{m_v} = \frac{m_d}{m_v}R$, where k is the Boltzmann constant and N is the Avogadro number, and (3.1) becomes $\frac{d\ln p}{dz} = -\frac{g}{(m_d/m_v)RT}$. Therefore (2.7) is rewritten as $\frac{L}{(m_d/m_v)R} \approx \frac{T^2}{p}\frac{dp}{dT} = -\frac{d\ln p}{d(1/T)} = \frac{g}{(m_d/m_v)RT}\frac{dz}{d(1/T)}$. $\therefore \frac{g}{LT} \approx \frac{d(1/T)}{dz} = -\frac{1}{T^2}\frac{dT}{dz}$

[12]It must be noted that this mechanism works only when the atmosphere is phase-changeable such as water, and does not work for major non-condensable greenhouse gases such as carbon dioxide.

[13]Because of $p \gg e_s$, the virtual temperature T_v defined in (2.6) may be identified approximately with T.

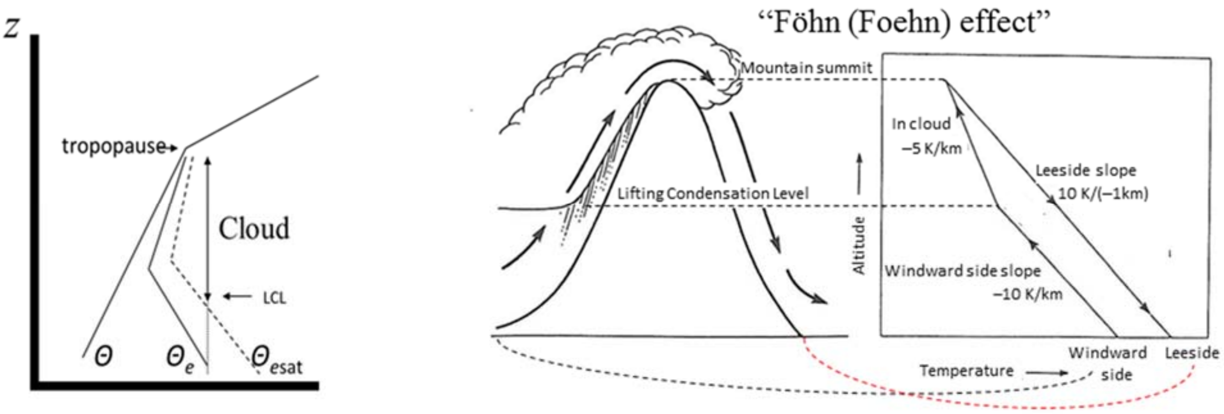

Fig. 3.6 Schematic picture of the tropical troposphere (left) and “Föhn (Foehn) effect” (right).

in the atmosphere. The latter makes latent heating, but products are omitted (as falling precipitation particles) immediately, which is called *pseudo-adiabatic*. From (2.5) and (2.4) with no external supply ($J = 0,\ S = 0$), we have

$$C_p \frac{D\ln\theta}{Dt} = -\frac{L}{T}\frac{Dr}{Dt}, \quad \therefore \ \frac{D\ln\theta_e}{Dt} = 0, \quad \theta_e \equiv \theta\cdot\exp\left(\frac{L}{C_p}\frac{r}{T}\right), \tag{3.10}$$

where θ_e is called *equivalent potential temperature*[14]. As the dry adiabatic lapse rate Γ is derived from θ =const., we may obtain *moist (pseudo-)adiabatic lapse rate* Γ_m from θ_e =const. and $r = r_s$ as follows:

$$0 = \frac{d\ln\theta_e}{dz} = -\frac{g}{RT}\frac{d\ln\theta_e}{d\ln p} = -\frac{g}{RT}\frac{d}{d\ln p}\left(\ln\theta + \frac{L}{C_p}\frac{r_s}{T}\right) \approx -\frac{g}{RT}\left(\frac{d\ln T}{d\ln p} - \frac{R}{C_p} + \frac{Lp}{C_p T}\frac{dr_s}{dp}\right)$$

$$\approx -\frac{g}{RT}\left[\frac{d\ln T}{d\ln p} - \frac{R}{C_p} + \frac{m_v}{m_d}\frac{pL}{C_p T}\frac{d}{dp}\left(\frac{e_s}{p}\right)\right] \approx -\frac{g}{RT}\left[\left(1 + \frac{m_v}{m_d}\frac{L^2 r_s}{C_p R T^2}\right)\frac{d\ln T}{d\ln p} - \frac{R}{C_p}\left(1 + \frac{L r_s}{RT}\right)\right]$$

$$\therefore \ \frac{d\ln T}{d\ln p} \approx \frac{R}{C_p}\cdot\frac{1 + \frac{L r_s}{RT}}{1 + \frac{(m_v/m_d)L^2 r_s}{C_p R T^2}}, \quad \therefore \ \frac{dT}{dz} \approx -\Gamma\frac{1 + \frac{L r_s}{RT}}{1 + \frac{(m_v/m_d)L^2 r_s}{C_p R T^2}} \equiv -\Gamma_m, \tag{3.11}$$

where we used (2.7) with approximations appropriate in the actual Earth’s atmosphere.

Substituting characteristic values, we have

$$\Gamma_s \text{ (about 1 K/km)} < \Gamma_m \text{ (about 5 K/km)} < \Gamma_c \text{ (about 6.5 K/km)} < \Gamma \text{ (about 10 K/km)}.$$

Thus the observed mean vertical temperature gradient Γ_c is stable ($\Gamma_c < \Gamma$; unlikely to generate convection) under the dry condition (fine weather without clouds), whereas it may be unstable ($\Gamma_c > \Gamma_m$; possible to generate convection) if condensation (or sublimation) occurs (inside clouds), which is called “conditional (convective) instability”. This contributes to make troposphere even though radiative equilibrium profile is barely stable for dry condition. However, as will be discussed in detail in Chapter 6, this is paradoxical, because clouds are most likely to be generated by upward flow associated with convection, but the convection is generated spontaneously only in the clouds. Even in the tropical troposphere any motion from the ground to the so-called lifting condensation level (LCL, where $r = r_s,\ \theta_e(r = r_s) = \theta_e(z = 0)$) must be forced (Fig. 3.6 left).

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[14] We assumed $\frac{1}{T}\frac{Dr}{Dt} = \frac{D}{Dt}\frac{r}{T} - r\frac{D}{Dt}\frac{1}{T} = \frac{D}{Dt}\frac{r}{T} + \frac{r}{T^2}\frac{DT}{Dt}$; $\frac{DT}{T} \ll \frac{Dr}{r}$. There have been several definitions/derivations for the equivalent potential temperature, but differences of their values are small in the actual Earth’s atmosphere.

Because of $\Gamma_m < \Gamma$, when an air flow passes beyond a mountain, the leeside becomes hotter and drier than the windward side (Fig. 3.6 right). This is known in Europe as "Föhn (Foehn) effect", and is considered also as a cause of forest fires in the El Niño phase (cf. Section 5.3).

3.4. Log-pressure coordinate

As have been shown derivations of several equations, $\ln p$ may be used as a vertical coordinate under the hydrostatic equilibrium (3.1). In observations, even after development of the GPS system, the altitude of a radiosonde balloon is calculated essentially by integration of (3.1). Substituting $\rho = p/RT$ from (2.6), (3.1) becomes $d\rho/\rho = -(g/RT_v)dz$, and we put the solution as ρ_0, namely

$$\rho_0(z) \propto \exp\left(-\int \frac{dz}{H}\right), \ H \equiv \frac{RT_v}{g} \ : \text{density scale height}, \tag{3.12}$$

and exchange the roles of variables p and z such as

$$dp/\rho_0 = gHd\ln p \ \rightarrow \ -gdz; \quad gdz \ \rightarrow \ d\phi,$$

where z re-defined newly by $\ln p$ is the "log-pressure" vertical coordinate (an independent variable) and ϕ defined by the geometric vertical coordinate z is called *geopotential* (a dependent variable corresponding to pressure). This is also a background of upper-air weather maps which are drawn by geopotential contour lines on pressure surfaces (standard pressure levels such as 850, 700, 500, 300, 200 and 100 hPa). The "vertical velocity" is re-defined by a pressure tendency as

$$-Hd\ln p/dt \rightarrow \ w,$$

so as to satisfy $w = dz/dt$ in the log-pressure coordinate[15].

Using the log-pressure coordinate, the hydrostatic equation (3.1) is written as

$$\frac{\partial \phi}{\partial z} = +\frac{RT_v}{H}, \tag{3.13}$$

which corresponds to an approximate form of the vertical component of the equation of motion (2.3). The other basic equations (the zonal and meridional components of (2.3), the continuity equation (2.4), and the thermodynamic equation (2.5)) are

$$\frac{Du}{Dt} - fv + \frac{\partial \phi}{\partial x} = F_x, \tag{3.14}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \phi}{\partial y} = F_y, \tag{3.15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial \rho_0 w}{\partial z} = 0, \tag{3.16}$$

$$\frac{DT}{Dt} + \Gamma w = \frac{J + Ls}{C_p}. \tag{3.17}$$

[15]In the "pressure coordinate" (used primarily in earlier numerical predictions) we may write $w = -(H/p)\omega$, where $\omega \equiv dp/dt$ (called "omega") is used for (downward) vertical velocity, and the continuity equation (3.16) is somewhat simpler as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$.

The log-pressure coordinate corresponds to a variety of (or mathematically equivalent with) the inelastic (or so-called Boussinesq) system[16] of fluid dynamics, in which the acoustic modes (sound waves) are omitted because the time derivative (of density) in the continuity equation is completely neglected. By considering the vertical gradient of temperature (corresponding essentially to density), the buoyancy modes (gravity waves, see Chapter 5) are included. These are completely appropriate for meteorology or atmospheric science in scales larger than mesoscales (say, 100 km) in the middle and high latitudes. These are valid also for general circulations and larger-scale waves in the tropics described in Chapters 4 and 5. However, for various local convections important in the tropics, the hydrostatic approximation (3.1) or (3.13) is not always appropriate, and we shall return to a complete form of the vertical component of equation of motion in Chapter 6.

Exercise 3

In the radiative equilibrium (as shown in the left-hand side panel of Fig. 3.2), the lower atmosphere temperature becomes hotter, if the atmosphere is thicker.

(1) If the atmosphere is very, very thick, the bottom temperature becomes hotter than the sun. Is it correct?

(2) There is a temperature gap between the lowest atmosphere and the ground. What appears as the result?

Answers:

(1) If the atmospheric temperature becomes too high, such as the sun, then the radiation is not infrared but visible (as the sun is radiating actually). So the assumption of gray body with infrared absorption and the radiative equilibrium as shown in the previous slide become invalid.

(2) Atmospheric motion or convection must be generated at least at this bottom, due to the convective instability of this too large temperature gradient. In other words the troposphere must be generated to relax this temperature gap.

[16]When contribution of ρ_0 in the continuity equation is neglected just as almost incompressible fluid (except only for buoyancy force in the hydrostatic equation), such a system is called Boussinesq approximation. In this meaning the "pressure coordinate" briefly mentioned in the previous footnote is close to a Boussinesq system.